TALKS FOR THE SCHOOL ON MOCK MODULAR FORMS AND RELATED TOPICS IN FUKUOKA, JAPAN

BEN KANE

Abstract. We investigate the modular completions of the mock theta functions.

1. Completions of the mock theta functions: from E_2 to Zwegers

1.1. **The Mock Theta functions.** In his last letter to Hardy, Ramanujan introduced a new family of functions which he dubbed "mock theta functions". He stated that the mock theta functions were not themselves theta functions, but that they behave like theta functions **Historical note:** in Ramanujan's time, "theta functions" was the terminology for "modular forms".

Recall the definition of modular forms: A function f is a modular form of weight k on $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ with multiplier system χ (having absolute value 1) if

(1) For all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, defining $f|_k M(\tau) = f|_{k,\chi} M(\tau) := \overline{\chi(M)}(c\tau+d)^{-k} f(M\tau)$ with $\tau \in \mathbb{H}$ (we omit χ when it is clear from the context)
$$f|_{k,\chi} M = f(\tau).$$
(2) The function f is holomorphic on $\mathbb{H}$.
(3) The function is holomorphic at the cusps: for every cusp $\rho = \frac{\alpha}{\gamma}$, the function $f_\rho := f|_k M_\rho^{-1}$, with $M_\rho = \left(\begin{smallmatrix} \alpha & * \\ \gamma & * \end{smallmatrix}\right)$, satisfies (here $q := e^{2\pi i\tau}$ and ℓ_ρ is the cusp form)
$$f_\rho(\tau) = \sum_{n\geq 0} a_{f,\rho}(n) q^{\frac{n}{\ell_\rho}}.$$

Ramanujan included 17 examples of his new mock theta functions, and listed some vague properties of the functions, but did not fully explain why they were like modular forms, nor did he prove that they were indeed not modular forms.

One of the main properties that he listed is the following: At each root of unity $\zeta = e^{2\pi i\rho}$ (with $\rho \in \mathbb{Q}$), and $q = e^{-t}\zeta$ with $t \to 0^+$ (i.e., approaching radially), there exist M, N such that the function has the asymptotic
$$\sum_{\mu=1}^{M} t^{k_\mu} \exp\left(\sum_{\nu=1}^{N} c_{\mu\nu} t^{\nu}\right) + O(1). \tag{1.1}$$

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In addition, he required that at each cusp, there is a modular form which cancels the growth at that cusp, and that there is not a single modular form which cancels the growth at all cusps simultaneously.

An example of a mock theta function given by Ramanujan (he calls it a mock theta function of order 3) is

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}.$$

G.N. Watson (1935) later proved that $f(q)$ indeed satisfies the asymptotic formula (1.1). In particular, he showed that

$$f(q) = \frac{1}{\prod_{\ell=1}^{\infty}(1-q^\ell)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{3}{2}n^2+\frac{1}{2}n}}{1+q^n}. \tag{1.2}$$

The denominator $\prod_{\ell=1}^{\infty}(1-q^\ell)$ is essentially the η-function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{\ell=1}^{\infty} \left(1-q^\ell\right).$$

Another mock theta function (of order 5) is given by

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)(1+q^2)\cdots(1+q^n)}.$$

Watson did not obtain a formula similar to (1.2) for f_0, but Andrews (1986) proved

$$f_0(q) = \frac{1}{\prod_{\ell=1}^{\infty}(1-q^\ell)} \sum_{n\geq 0} \sum_{|j|\leq n} (-1)^j q^{\frac{5}{2}n^2+\frac{1}{2}n-j^2} \left(1-q^{4n+2}\right).$$

Question: How does this fit into the theory of modular forms?
"Rough" answer (Zwegers): It is part of a "modular object".
What do we mean by that?

1.2. **Completions: a simple example.** Let's start with a simple example:
Consider the Eisenstein series

$$E_k(\tau) := \sum_{(c,d)=1} (c\tau+d)^{-k}.$$

Converges absolutely and locally uniformly for $k > 2$. Note that (here $\Gamma_\infty := \{\pm T^n : n \in \mathbb{Z}\}$ with $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$)

$$E_k = \sum_{M\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} 1|_k M.$$

For $2 < k \in 2\mathbb{N}$, when slashing with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we may use $|_k M|_k \gamma = |_k (M\gamma)$ ot obtain that E_k is a modular form of weight k. What about $k = 2$?

As is well-known, if one writes the Eisenstein series

$$E_k(\tau) = \sum_c \sum_{d \pmod c}^{*} \sum_{n\in\mathbb{Z}} (c\tau+d+nc)^{-k},$$

then one obtains the Fourier expansion (B_k is kth Bernoulli number and $\sigma_\ell(n) := \sum_{d|n} d^\ell$ is the sum of divisors)

$$E_k(\tau) = 1 + \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n.$$

Similarly, by choosing the same ordering on the conditionally convergent series defining E_2, we obtain ($\sigma = \sigma_1$)

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n.$$

Although E_2 is holomorphic, it is not modular of weight 2 (this is a consequence of the fact that the sum is only conditionally convergent, and hence reordering is not possible). However, it does satisfy a transformation property:

$$E_2(M\tau) = (c\tau+d)^2E_2(\tau) - \frac{6i}{\pi}c(c\tau+d) \tag{1.3}$$

Idea: Find something else with the same transformation property as (1.3). Note that ($\tau = u + iv$ throughout)

$$\begin{aligned}\frac{1}{\mathrm{Im}(M\tau)} = \frac{|c\tau+d|^2}{v} &= \frac{(c\tau+d)(c\overline{\tau}+d)}{v} \\ &= \frac{(c\tau+d)(c(\tau-2iv)+d)}{v} = \frac{(c\tau+d)^2}{v} - 2ic(c\tau+d).\end{aligned}$$

Hence

$$E_2^*(\tau) := E_2(\tau) - \frac{3}{\pi v}$$

is a non-holomorphic modular form and we conclude that E_2 is "part" of a non-holomorphic modular form, i.e., the holomorphic function E_2 has been *completed* to a non-holomorphic modular form. This is the type of behavior meant above by "part of a modular object".

1.3. **Appell–Lerch sums and their completions.** To understand how Ramanujan's mock theta functions are like E_2, we first need to understand sums like (1.2). We begin by defining Jacobi's theta functions

$$\vartheta(z;\tau) := \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} e^{\pi i\nu^2\tau + 2\pi i\nu\left(z+\frac{1}{2}\right)} = \sum_{n\in\frac{1}{2}+\mathbb{Z}} (-1)^n q^{\frac{n^2}{2}}\zeta^n,$$

where $\zeta := e^{2\pi iz}$. A normalized Appell–Lerch sum is defined by

$$\mu(u,v) = \mu(u,v;\tau) := \frac{e^{\pi iu}}{\vartheta(v;\tau)}\sum_{n\in\mathbb{Z}}\frac{(-1)^n e^{\pi i\left(n^2+n\right)\tau+2\pi inv}}{1-e^{2\pi in\tau+2\pi iu}}.$$

Note: The Jacobi triple product formula states that

$$\vartheta(z;\tau) = -iq^{\frac{1}{8}}\zeta^{-\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^n)\left(1-\zeta q^{n-1}\right)\left(1-\zeta^{-1}q^n\right).$$

The μ-function satisfies the following properties.

Proposition 1.1 (Lerch/Zwegers)**.**

(1)
$$\mu(u+1,v) = -\mu(u,v).$$

(2)
$$\mu(u,v+1) = -\mu(u,v)$$

(3)
$$\mu(u,v) + e^{-2\pi i(u-v)-\pi i\tau}\mu(u+\tau,v) = -e^{-\pi i(u-v)-\pi i\tau/4}.$$

(4)
$$\mu(u+\tau,v+\tau) = \mu(u,v)$$

(5)
$$\mu(-u,-v) = \mu(u,v)$$

(6) The function $u \mapsto \mu(u,v)$ is meromorphic with simple poles at $u = n\tau + m$ ($n,m \in \mathbb{Z}$), and particularly has residue $-\frac{1}{2\pi i\vartheta(v;\tau)}$ at $u=0$.

(7)
$$\mu(u+z,v+z) - \mu(u,v) = \frac{1}{2\pi i}\frac{\frac{\partial\vartheta}{\partial w}(w;\tau)\big|_{w=0}\vartheta(u+v+z;\tau)\vartheta(z;\tau)}{\vartheta(u;\tau)\vartheta(v;\tau)\vartheta(u+z;\tau)\vartheta(v+z;\tau)}.$$

(8)
$$\mu(v,u) = \mu(u,v).$$

(9) The function μ has the transformation properties
$$\mu(u,v;\tau+1) = e^{-\frac{\pi i}{4}}\mu(u,v;\tau).$$
and
$$\frac{1}{\sqrt{i\tau}}e^{\pi i\frac{(u-v)^2}{\tau}}\mu\left(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}\right) + \mu(u,v;\tau) = \frac{1}{2i}h(u-v;\tau),$$
where
$$h(z;\tau) := \int_{\mathbb{R}}\frac{e^{\pi i\tau x^2-2\pi izx}}{\cosh(\pi x)}dx.$$

The last two properties are somewhat reminiscent of the transformation properties of E_2. If we can find another function (similar to $-\frac{3}{\pi y}$ in the case of E_2) which misses modularity in the same way as μ, then we can "complete" it to obtain a "modular object".

We next show how to complete the Lerch sums. Define
$$E(z) := 2\int_0^z e^{-\pi u^2}du = \sum_{n=0}^{\infty}\frac{(-\pi)^n}{n!}\frac{z^{2n+1}}{n+\frac{1}{2}} = \operatorname{sgn}(z)\left(1-\beta(z^2)\right),$$
with
$$\beta(x) := \int_x^{\infty}u^{-\frac{1}{2}}e^{-\pi u}du.$$
For $u \in \mathbb{C}$ and $\tau \in \mathbb{H}$ we then define (set $a = \frac{\operatorname{Im}(u)}{\operatorname{Im}(\tau)}$)
$$R(u) = R(u;\tau) := \sum_{n\in\frac{1}{2}+\mathbb{Z}}\left(\operatorname{sgn}(v) - E\left((n+a)\sqrt{2v}\right)\right)(-1)^{n-\frac{1}{2}}e^{-\pi in^2\tau-2\pi inu}.$$

Note that because of the appearance of a and v, the function $R(u;\tau)$ is non-holomorphic, both as a function of u and of τ, but it is real analytic. Indeed, its image under differentiation with respect to $\overline{u}$ essentially gives one of Jacobi's theta functions. In particular,

$$\frac{\partial R}{\partial \overline{u}}(u;\tau) = \sqrt{\frac{2}{v}} e^{-2\pi a^2 v} \vartheta(\overline{u}; -\overline{\tau})$$

and

$$\frac{\partial R}{\partial \overline{\tau}}(a\tau - b;\tau) = -\frac{i}{\sqrt{2v}} e^{-2\pi a^2 v} \sum_{n\in\frac{1}{2}+\mathbb{Z}} (-1)^{n-\frac{1}{2}} (n+a) e^{-\pi i n^2 \overline{\tau} - 2\pi i n(a\overline{\tau} - b)}.$$

The right-hand side is essentially a weight 3/2 unary theta function. Define

$$\xi_\kappa = \xi_{\kappa,\tau} := 2iv^\kappa \overline{\frac{\partial}{\partial \overline{\tau}}},$$

which maps functions satisfying weight κ modularity to functions satisfying weight $2-\kappa$ modularity (consider simultaneous modularity in $(\tau,\overline{\tau})$, then for f satisfying weight $(\kappa,0)$ modularity, $\frac{\partial}{\partial\overline{\tau}}(f)$ satisfies weight $(\kappa,2)$ modularity, conjugating gives weight $(2,\kappa)$, and then multiplying by the weight $(-\kappa,-\kappa)$ form v^κ yields a weight $(2-\kappa,0)$ form). Then ($\doteq$ means up to non-zero constant) for $\alpha \in (-1/2, 1/2)$ and $\beta \in \mathbb{R}$ we have

$$\xi_{1/2,\tau}\left(R(\alpha\tau - \beta;\tau)\right) \doteq g_{\alpha+\frac{1}{2},\beta+\frac{1}{2}}(\tau), \tag{1.4}$$

where

$$g_{a,b}(\tau) := \sum_{n\in a+\mathbb{Z}} n q^{\frac{n^2}{2}} \zeta^{bn}$$

The function R satisfies certain transformation properties.

Proposition 1.2.
First, it satisfies certain elliptic transformation properties:

(1)
$$R(u+1) = -R(u).$$

(2)
$$R(u) + e^{-2\pi i u - \pi i \tau} R(u+\tau) = 2e^{-\pi i u - \frac{\pi i \tau}{4}}.$$

(3)
$$R(-u) = -R(u).$$

It also satisfies modular transformation properties:

(1)
$$R(u;\tau+1) = e^{-\frac{\pi i}{4}} R(u;\tau).$$

(2)
$$\frac{1}{\sqrt{-i\tau}} e^{\pi i \frac{u^2}{\tau}} R\left(\frac{u}{\tau}; -\frac{1}{\tau}\right) + R(u;\tau) = h(u;\tau).$$

Comparing Proposition 1.2 with Proposition 1.1, we see that the functions essentially "miss modularity" in the same way. It is thus natural to define

$$\widetilde{\mu}(u,v;\tau) := \mu(u,v;\tau) + \frac{i}{2} R(u-v;\tau).$$

Theorem 1.3. *The function $\widetilde{\mu}$ is real analytic (but not holomorphic) and satisfies the following modularity properties:*

(1) The function $\widetilde{\mu}$ satisfies the elliptic transformation property

$$\widetilde{\mu}(u+k\tau+\ell, v+m\tau+n) = (-1)^{k+\ell+m+n} e^{\pi i(k-m)^2\tau+2\pi i(k-m)(u-v)}\widetilde{\mu}(u,v)$$

(2) Let χ be the η-multiplier (i.e., $\chi(M) := \frac{\eta(M\tau)}{(c\tau+d)^{\frac{1}{2}}\eta(\tau)}$). Then for every $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the function $\widetilde{\mu}$ satisfies the modular transformation property

$$\widetilde{\mu}\left(\frac{u}{c\tau+d}, \frac{v}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = \chi(M)^{-3}(c\tau+d)^{\frac{1}{2}} e^{-\pi i \frac{c(u-v)^2}{c\tau+d}}\widetilde{\mu}(u,v;\tau).$$

(3)

$$\widetilde{\mu}(-u,-v) = \widetilde{\mu}(v,u) = \widetilde{\mu}(u,v).$$

(4) We have

$$\widetilde{\mu}(u+z,v+z) - \widetilde{\mu}(u,v) = \frac{1}{2\pi i}\frac{\frac{\partial\vartheta}{\partial w}(w;\tau)\big|_{w=0}\vartheta(u+v+z;\tau)\vartheta(z;\tau)}{\vartheta(u;\tau)\vartheta(v;\tau)\vartheta(u+z;\tau)\vartheta(v+z;\tau)}.$$

Remark. There are important properties of the functions μ, R, and $\widetilde{\mu}$ which closely resemble the properties of E_2, $\frac{1}{y}$, and E_2^*. Namely, the first is meromorphic, the second is "simpler", and the third satisfies modularity properties.

We see that μ is like E_2 and $\widetilde{\mu}$ is like E_2^*. However, it also has elliptic transformation properties like the Weierstrass elliptic functions (e.g., the Weierstrass $\wp$-function). Two variable functions which satisfy both modularity and elliptic transformations are what are known as *Jacobi forms*. More precisely, a Jacobi form of weight k and index m is a holomorphic function $\phi : \mathbb{H}\times\mathbb{C}\to\mathbb{C}$ satisfying the transformation properties

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi i mcz^2}{c\tau+d}}\phi(\tau,z)$$

and

$$\phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m\left(\lambda^2\tau+2\lambda z\right)}\phi(\tau,z).$$

Jacobi forms are so-named because the Jacobi theta functions $\vartheta(z;\tau)$ are Jacobi forms of weight 1/2 and index 1/2.

Theorem 1.3 shows that $\widetilde{\mu}$ has similar properties to those of a weight 1/2 Jacobi form, except that it is a vector-valued analogue with index $\left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ (or index 1 with $z = u - v$).

One can then use formulas like (1.2) to realize $f(q)$ as a certain μ-function. Hence $f(q)$ is the "holomorphic part" μ of a weight 1/2 non-holomorphic function $\widetilde{\mu}$. Namely,

$$\frac{q^{-\frac{1}{24}}}{2}f(q) = \frac{\eta^3(3\tau)}{\eta(\tau)\vartheta\left(\frac{3}{2};3\tau\right)} + q^{-\frac{1}{6}}\mu\left(\frac{3}{2},-\tau;3\tau\right) - q^{-\frac{1}{6}}\mu\left(\frac{3}{2},\tau;3\tau\right).$$

The non-holomorphic piece R that needed to be added to $f(q)$ is related to a weight 3/2 unary theta function. The non-holomorphic part may be realized as what is known as a *non-holomorphic Eichler integral* (also known as a period integral)

$$e^{-\pi i a^2\tau+2\pi i\left(b+\frac{1}{2}\right)}R(a\tau-b) = \int_{-\overline{\tau}}^{i\infty}\frac{g_{a+\frac{1}{2},b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}dz$$

of the weight 3/2 unary theta function $g_{a+1/2,b+1/2}$.

2. Fourier coefficients of harmonic Maass forms and weakly holomorphic modular forms with applications

2.1. **Weakly holomorphic modular forms and harmonic Maass forms.** In the last lecture, we investigated how the mock theta functions are "part" of a "modular object". We now describe more precisely what we mean.

Recall the definition of modular forms: A function f is a modular form of weight k on $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ with multiplier system χ (having absolute value 1) if

(1) For all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, defining $f|_k M(\tau) := \overline{\chi(M)}(c\tau+d)^{-k} f(M\tau)$ with $\tau \in \mathbb{H}$
$$f|_k M = f(\tau).$$
(2) The function f is holomorphic on $\mathbb{H}$.
(3) The function is holomorphic at the cusps: for every cusp $\rho = \frac{\alpha}{\gamma}$, the function $f_\rho := f|_k M_\rho^{-1}$, with $M_\rho = \left(\begin{smallmatrix} \alpha & * \\ \gamma & * \end{smallmatrix}\right)$, satisfies (here $q := e^{2\pi i\tau}$ and ℓ_ρ is the cusp width)
$$f_\rho(\tau) = \sum_{n\geq 0} a_{f,\rho}(n) q^{\frac{n}{\ell_\rho}}.$$

The main aspect of the definition is the modularity property, and it is natural to relax the other conditions. For example, we may relax the third condition by allowing a finite-order pole at the cusps (i.e., the Laurent expansion in q is supported in finitely many negative powers of q). In other words, these are meromorphic modular forms whose only poles occur at the cusps, and such functions are known as *weakly holomorphic modular forms*. Of course, if the condition that f is holomorphic on $\mathbb{H}$ is also relaxed to allow finite-order poles in the upper half-plane, one obtains the space of meromorphic modular forms.

Recalling that Zwegers's μ-function is part of a non-holomorphic modular form, one can try to classify such functions by placing them into a more general framework. For this, we need a natural family of non-holomorphic modular forms. Zwegers's functions are rather smooth, so it seems natural to require them to be real analytic. Recall that μ is holomorphic in τ and $\xi_{1/2}$ sends R to a weight $3/2$ unary theta function. Thus $\xi_{1/2}$ sends $\widehat{\mu}$ to a weight $3/2$ unary theta function. Since the weight $3/2$ unary theta function is holomorphic, it is annihilated by $\frac{\partial}{\partial\overline{\tau}}$, and hence by $\xi_{3/2}$, in particular. Similarly,
$$\xi_2\left(E_2^*(\tau)\right) = -\frac{3}{\pi}\xi_2\left(\frac{1}{y}\right) = -\frac{3}{\pi}2iv^2\overline{\frac{i}{2}\frac{\partial}{\partial v}\left(\frac{1}{v}\right)} = \frac{3}{\pi}.$$
Hence $\xi_0 \circ \xi_2\left(E_2^*\right) = 0$. These examples are part of a more general family of non-holomorphic modular forms satisfying a certain differential equation. For $\kappa \in \frac{1}{2}\mathbb{Z}$, define the *weight κ hyperbolic Laplacian*
$$\Delta_\kappa := -v^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial v}\right) = -\xi_{2-\kappa} \circ \xi_\kappa.$$
Then a *harmonic Maass form* of weight κ on Γ is a real-analytic function $\mathcal{F} : \mathbb{H} \to \mathbb{C}$ satisfying the following.

(1) We have $\mathcal{F}|_\kappa M = \mathcal{F}$ for every $M \in \Gamma$.
(2) The function $\mathcal{F}$ is annihilated by Δ_κ (i.e., $\Delta_\kappa(\mathcal{F}) = 0$).
(3) The function $\mathcal{F}$ has at most linear exponential growth at all cusps of Γ.

If the second condition is replaced with $\Delta_\kappa(\mathcal{F}) = \lambda\mathcal{F}$, then we call $\mathcal{F}$ a *weak Maass form* with eigenvalue λ.

Assuming that Γ is a congruence subgroup, a harmonic Maass form $\mathcal{F}$ has a Fourier expansion

$$\mathcal{F}(\tau) = \sum_{n\in\mathbb{Z}} a_{\mathcal{F},v}(n) q^{\frac{n}{\ell_\infty}}.$$

We assume without loss of generality that $\ell_\infty = 1$. This Fourier expansion has a natural splitting into a *holomorphic part* $\mathcal{F}^+$ and a non-holomorphic part $\mathcal{F}^-$, so that

$$\mathcal{F} = \mathcal{F}^+ + \mathcal{F}^-.$$

Since there are two independent solutions to the second-order differential equation $\Delta_\kappa(\mathcal{F}) = 0$, for $\kappa < 1$ one can write

$$\begin{aligned}\mathcal{F}^+(\tau) &= \sum_{n\gg-\infty} a_{\mathcal{F}}^+(n) q^n\\ \mathcal{F}^-(\tau) &= a_{\mathcal{F}}^-(0) v^{1-\kappa} + \sum_{\substack{n\ll\infty\\ n\neq 0}} a_{\mathcal{F}}^-(n)\Gamma(1-\kappa, -4\pi n y) q^n,\end{aligned}$$

where $\Gamma(s,y) := \int_y^\infty t^{s-1}e^{-t}dt$ is the (upper) incomplete gamma function. The holomorphic part is often called a *mock modular form.*

The kernel of ξ_κ is meromorphic forms, so in some sense harmonic Maass forms give a natural second-order extension of meromorphic modular forms. Indeed, if $\mathcal{F}$ is a harmonic Maass form, then $\xi_\kappa(\mathcal{F})$ is annihilated by $\xi_{2-\kappa}$, and is hence meromorphic. The fact that $\mathcal{F}$ is real analytic on $\mathbb{H}$ implies that $\xi_\kappa(\mathcal{F})$ is furthermore weakly holomorphic. We saw in the last lecture that $\xi_\kappa(\mathcal{F})$ satisfies weight $2-\kappa$ modularity, so it is thus a weight $2-\kappa$ weakly holomorphic modular form. If one is given only a mock modular form (such as one of Ramanujan's mock theta functions), then one calls the resulting weight $2-\kappa$ form the *shadow* of the mock modular form (since it is "hidden" if one is only given the mock modular form). The shadows of the mock theta functions were all unary theta functions.

2.2. **Growth of the coefficients.** The coefficients of negative-weight weakly holomorphic modular forms and mock modular forms have been studied by a number of authors. Consider first a well-known example given by the partition function. Let $p(n)$ denote the number of partitions of an integer n (i.e., the number of distinct ways to write n as a sum of positive integers, where the order is irrelevant). Then the generating function is

$$P(z) := 1 + \sum_{n=1}^\infty p(n)q^n = \frac{1}{\prod_{n=1}^\infty(1-q^n)},$$

which is essentially the weight $-1/2$ weakly holomorphic modular form $\frac{1}{\eta}$. Hardy and Ramanujan developed the Circle Method to obtain an asymptotic formula for $p(n)$. Note that by the Residue Theorem

$$\frac{1}{2\pi i}\int P(z) q^{-n-1} dq = p(n),$$

where the integral is any contour around zero. In the Circle Method, one splits this integral into major and minor arcs in order to obtain the main asymptotic growth of $p(n)$ and an error term. Doing so, Hardy and Ramanujan obtained

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

Rademacher refined Hardy and Ramanujan's methods by choosing a better contour for which he could obtain better bounds. Rademacher then obtained a sequence of bounds with increasing accuracy (i.e., with decaying error term) and ultimately obtained what is known as an *exact formula* for $p(n)$. Namely, letting $I_s(x)$ denote the I-Bessel function and defining the Kloosterman sum

$$A_c(n) := \frac{1}{2}\sqrt{\frac{c}{12}} \sum_{\substack{x \pmod{24c} \\ x^2 \equiv -24n+1 \pmod{24c}}} \chi_{12}(x) e^{\frac{\pi i x}{6c}},$$

Rademacher showed that

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6c}\right).$$

Rademacher and Zuckerman then later showed similar exact formulas for all negative-weight weakly holomorphic modular forms. Roughly speaking, they showed that the nth coefficient of a weight $\kappa < 0$ weakly holomorphic modular form is a sum of Kloosterman sums times I-Bessel functions.

Knopp then investigated a sort of converse question for this. The question that he addressed was whether any function whose Fourier coefficients had the same shape as Rademacher's formula were always the coefficients of a modular form. Knopp answered this converse question in the negative. Namely, he found certain examples of forms which missed modularity in a "predictable way" (they are period polynomials) which have formulas with the same shape as Rademacher and Zuckerman's formulas for negative weight weakly holomorphic modular forms. It turns out that Knopp had uncovered families of (integral weight) mock modular forms, but their connections to Ramanujan's mock theta functions would not be discovered for many years. Namely, the completions of the functions investigated by Knopp were not known at the time.

2.3. **Formulas for the coefficients of the mock theta functions.** Recall the third-order mock theta function

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2 \cdots (1+q^n)^2} = 1 + \sum_{n=1}^{\infty} \alpha(n) q^n.$$

Ramanujan claimed that

$$\alpha(n) = \frac{(-1)^{n-1}\sqrt{6}}{\sqrt{24n-1}} e^{\pi\sqrt{\frac{n}{6}-\frac{1}{144}}} + O\left(\frac{e^{\frac{\pi}{2}\sqrt{\frac{n}{6}-\frac{1}{144}}}}{\sqrt{24n-1}}\right)$$

Dragonette proved this claim in her Ph.D. thesis and Andrews extended her work in his Ph.D. thesis to obtain the asymptotic formula

$$\alpha(n) = \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{c=1}^{\lfloor\sqrt{n}\rfloor} \frac{A_{2c}\left(n - \frac{c(1+(-1)^c)}{4}\right)}{c} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12c}\right) + O(n^{\varepsilon}) .$$

Noting the similarity to the exact formula for the partition function, the Andrews–Dragonette Conjecture claimed that

$$\alpha(n) = \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{c=1}^{\infty} \frac{A_{2c}\left(n - \frac{c(1+(-1)^c)}{4}\right)}{c} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12c}\right) . \tag{2.1}$$

However, since $f(q)$ is not modular, the techniques used by Rademacher and Zuckerman did not yield a sufficiently small error term to allow the sum to go to infinity.

Following Zwegers's realization of the mock theta functions as part of a modular object (namely, as the holomorphic part of a harmonic Maass form), Bringmann and Ono realized that the modularity properties could be used to obtain the exact formula (2.1).

2.4. **Poincaré series and Bringmann–Ono.** Instead of arguing via the Circle Method, Bringmann and Ono employed Poincaré series to obtain their exact formula. A *Poincaré series* is formed by taking a test function $\phi : \mathbb{H} \to \mathbb{C}$ and defining

$$\mathcal{P}_{\kappa,\Gamma,\phi} := \sum_{M \in \Gamma_\infty \backslash \Gamma} \phi|_\kappa M.$$

Whenever the series converges absolutely and locally uniformly, The function $\mathcal{P}_{\kappa,\Gamma,\phi}$ satisfies weight κ modularity on Γ. The choice $\phi(\tau) := 1$ yields the Eisenstein series, which is part of a family of weakly holomorphic modular forms $P_{\kappa,m}$ given by taking $\phi(\tau) = \phi_m(\tau) := e^{2\pi i m\tau}$

Now set

$$\mathcal{M}_{\kappa,s}(w) := |w|^{-\frac{\kappa}{2}} M_{\mathrm{sgn}(w)\frac{\kappa}{2}, s-\frac{1}{2}}(|w|),$$

where $M_{\nu,\mu}$ is the M-Whittaker function and choose (recall that $\tau = u + iv$)

$$\varphi_{\kappa,m,s}(\tau) := \mathcal{M}_{\kappa,s}(4\pi m v) e^{2\pi i m u}.$$

One can check that $\varphi_{\kappa,m,s}$ is an eigenfunction under Δ_κ with eigenvalue $(s - \kappa/2)(1 - s - \kappa/2)$. Hence the Poincaré series

$$\mathcal{P}_{\kappa,m,\Gamma,s} := \sum_{M \in \Gamma_\infty \backslash \Gamma} \varphi_{\kappa,m,s}|_\kappa M$$

taken from choosing $\phi = \varphi_{\kappa,m,s}$ is also an eigenfunction, as long as it converges absolutely and locally uniformly. In particular, choosing $s = \frac{\kappa}{2}$ or $s = 1 - \frac{\kappa}{2}$ yields a harmonic Maass form $\mathcal{P}_{\kappa,m}$. The series converges absolutely and locally uniformly for $\mathrm{Re}(s) > 1$, and in particular for $\kappa < 0$ or $\kappa > 2$ we obtain a harmonic Maass form.

A key property about harmonic Maass forms is the fact that if they do not exhibit growth towards the cusps, then they must be holomorphic modular forms; this is a result of Niebur, who studied the weight zero version of these Poincaré series. Thus for weight < 0 they must exhibit growth towards the cusps. In other words, a negative-weight harmonic Maass form is uniquely determined by the terms in its Fourier expansion $\mathcal{F}^+ + \mathcal{F}^-$ which exhibit growth. The sum of such terms is known as the *principal part* of $\mathcal{F}$. In other words, the principal part is the terms in $\mathcal{F}^+$ with $n < 0$ or in $\mathcal{F}^-$ with $n \geq 0$. For $\kappa < 0$, we see that the Poincaré series $\mathcal{P}_{\kappa,m}$ then form a basis of the space of harmonic Maass forms of weight κ (on Γ).

The principal part of each $\mathcal{P}_{\kappa,m}$ is given by the term coming from the identity matrix, and is hence essentially $\varphi_{\kappa,m,1-\kappa/2}$. We may further split $\varphi_{\kappa,m,1-\kappa/2}$ naturally into its holomorphic and non-holomorphic parts to obtain the following.

Proposition 2.1. *Suppose that $\kappa < 0$. For $m < 0$, the principal part of $\mathcal{P}_{\kappa,m}$ is a constant multiple of q^m. For $m = 0$, the principal part is a constant multiple of $v^{1-\kappa}$, and for $m > 0$, the principal part is a constant multiple of $\Gamma(1-\kappa, -4\pi m v)q^m$.*

The idea now to prove the Andrews–Dragonette Conjecture is to compute the coefficients of the Poincaré series and then write $f(q)$ as an explicit linear combination of the Poincaré series. One first needs to determine the group Γ under consideration. Zwegers had investigated the completion and modularity properties of $f(q)$ as part of a vector-valued modular form. As a consequence of his results, one obtains the following:

Theorem 2.2 (Zwegers, Bringmann–Ono)**.** *The function $q^{-1}f(q^{24})$ is the holomorphic part of a weight $1/2$ harmonic Maass form on $\Gamma_0(144)$, where as usual we have $\Gamma_0(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : N \mid c\}$. Moreover, it has a simple pole at $i\infty$ and no other singularities.*

Hence we are interested in taking $\Gamma = \Gamma_0(N)$. By Theorem 2.2 and the fact that the principal part uniquely determines the form up to holomorphic modular forms, one may use Proposition 2.1 to guess that $q^{-1}f(q^{24})$ is a constant multiple of $\mathcal{P}_{1/2,-1}$. It would then seem that the question is resolved after computing the Fourier expansion of $\mathcal{P}_{1/2,-1}$. However, the Poincaré series does not converge absolutely in this case, so some care is needed. In order to address this issue, one computes the Fourier expansion of $\mathcal{P}_{\kappa,\Gamma_0(N),m,s}$ for $\mathrm{Re}(s) > 1$ and then analytically continues to $s = 3/4$.

By rewriting the sum over $\Gamma_\infty \backslash \Gamma_0(N)$ as a sum over $(c,d) = 1$ (like the Eisenstein series) and writing the inner sum in d (assuming that we have absolute convergence), we may compute the Fourier expansion by Poisson summation,

$$\sum_{n\in\mathbb{Z}} g(n) = \sum_{n\in\mathbb{Z}} \widehat{g}(n),$$

where $\widehat{g}$ is the Fourier transform. Assume that $m < 0$. Then one obtains that the coefficient of q^n have the same shape as (2.1) and the coefficient of $\Gamma(1-\kappa, -4\pi n v)q^n$ are essentially the same, except that the I-Bessel function has been replaced with the J-Bessel function. More generally, for $\mathrm{Re}(s) > 1$, the parameter of the Bessel functions is replaced with $2s - 1$. Using bounds for the Kloosterman sums, one can show that the Fourier expansion may be analytically continued beyond $s = 3/4$, yielding the Andrews–Dragonette conjecture.

Applications of Fourier coefficients of mock modular forms do not end there. For example, one may interpret $f(q)$ as the generating function for a certain partition-theoretic invariant, and further examples of this invariant yield more mock modular forms. After Ramanujan famously found the congruences for the partition function

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned}$$

in 1944, Dyson (an undergraduate at the time) tried to find a "combinatorial" explanation for these congruences. He defined a statistic known as the *rank* of a partition, which is defined to be the size of the largest part minus the number of parts. He conjectured that for the first 2 congruences, the rank modulo 5 (resp. modulo 7) equally splits the partition into 5 (resp. 7) distinct groups each of the same size (thus giving an explanation why the size is divisible by 5 (resp. 7). This was later proven by Atkin and Swinnerton-Dyer. Dyson also proposed that there was a similar statistic (that he called the "crank") which explained all 3 congruences; this was later found by Andrews and Garvan. Let $N(m,n)$ denote the number of partitions of n with rank m. Then the *rank generating function* is given by

$$R(w;q) := 1 + \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n) w^m q^n = \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{\ell=1}^{n}\left(1-wq^{\ell}\right)\left(1-w^{-1}q^{\ell}\right)}$$

Plugging in $w=1$, we get

$$R(1;q) = P(z),$$

while $w=-1$ yields

$$R(-1;q) = f(q).$$

In other words, $f(q)$ is a generating function for the number of partitions of n with even rank minus the number of partitions with odd rank. It turns out that this is not a coincidence.

Theorem 2.3 (Bringmann–Ono)**.** *If ζ is a root of unity, then $R(\zeta;q)$ is the holomorphic part of a weight $1/2$ harmonic Maass form.*

Bringmann and Ono gave the specific completion of $R(\zeta;q)$ and modularity properties of the completion.

As a consequence of their result, let $N(r,t;n)$ be the number of partitions of n with rank congruent to r modulo t.

Theorem 2.4 (Bringmann–Ono)**.** *The generating function ($\ell_t := \operatorname{lcm}(2c^2, 24)$)*

$$\sum_{n=0}^{\infty}\left(N(r,t;n) - \frac{p(n)}{t}\right) q^{\ell_t\left(n-\frac{1}{24}\right)}$$

is the holomorphic part of a weight $1/2$ harmonic Maass form.

Theorem 2.4 has a number of interesting consequences. Firstly, For $0 \le r < s < t$ we may consider

$$\sum_{n=0}^{\infty}\left(N(r,t;n) - N(s,t;n)\right) q^{\ell_t\left(n-\frac{1}{24}\right)}.$$

This is also the holomorphic part of a weight $1/2$ harmonic Maass form. There is a natural symmetry of partitions which implies that $N(r,t;n) = N(t-r,t;n)$, so one can also restrict $0 \leq r < s \leq \frac{t}{2}$. By investigating the asymptotics of the Fourier coefficients, one can then determine inequalities like $N(r,t;n) > N(s,t;n)$ for n sufficiently large. This leads to the resolution of a conjecture of Andrews and Lewis (up to a finite number of exceptions where equality holds).

Theorem 2.5 (Bringmann)**.** *For $n > 0$, we have*

$$N(0,3;n) < N(1,3;n) \qquad \text{if } n \equiv 0,2 \pmod 3,$$
$$N(0,3;n) > N(1,3;n) \qquad \text{if } n \equiv 1 \pmod 3.$$

This is part of a more general phenomenon.

Theorem 2.6 (Bringmann–K.)**.** *Assume that $c > 9$ is an odd integer. If $0 \leq a < b \leq \frac{c-1}{2}$, then for n sufficiently large (depending on a, b, c), we have*

$$N(a,c;n) > N(b,c;n).$$

Furthermore, if $c = 3, 5$, or 7, then for n sufficiently large, the inequality ($>$ or $<$) is completely determined by a, b, and the congruence class of n modulo c.

As another application, the non-holomorphic part is closely related to a unary theta function, and hence is only supported on certain square classes. For a Fourier expansion $\sum_{n\in\mathbb{Z}} a_v(n)q^n$, consider the "sieved" function

$$\sum_{\substack{n\in\mathbb{Z} \\ n\equiv A \pmod B}} a_v(n)q^n.$$

Then for A and B chosen appropriately so that the non-holomorphic part vanishes, the resulting function is weakly holomorphic. Since weakly holomorphic modular forms with integral coefficients have bounded denominators, this has algebraic applications.

Theorem 2.7 (Bringmann–Ono)**.** *Let t be a positive odd integer, and let $Q \nmid 6t$ be prime. If j is a positive integer, then there are infinitely many non-nested arithmetic progressions $An + B$ such that for every $0 \leq r < t$ we have*

$$N(r,t;An+B) \equiv 0 \pmod{Q^j}.$$

3. Fourier coefficients of polar harmonic Maass forms and meromorphic modular forms

3.1. **Ramanujan–like expansions.** Instead of negative-weight weakly holomorphic modular forms like $1/\eta$ related to the partition function, we investigate here coefficients of meromorphic modular forms. We relax the holomorphicity condition of modular forms. A function f is a meromorphic modular form of weight k on $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ with multiplier system χ (having absolute value 1) if

(1) For all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, defining $f|_k M(\tau) = f|_{k,\chi} M(\tau) := \overline{\chi(M)}(c\tau+d)^{-k} f(M\tau)$ with $\tau \in \mathbb{H}$ (we omit χ when it is clear from the context)
$$f|_{k,\chi} M = f(\tau).$$
(2) The function f is meromorphic on $\mathbb{H}$.
(3) The function is meromorphic at the cusps:
$$f_\sigma(\tau) = \sum_{n \gg -\infty} a_{f,\sigma}(n) q^{\frac{n}{\ell_\sigma}}.$$

If the growth at the cusps instead satisfies $v^{\kappa/2} f(\tau)$ vanishes (i.e., it grows like that of a cusp form towards cusps), then we call it a *meromorphic cusp form.*

Consider the example $1/E_4$. The coefficients grow **very** fast:
$$\frac{1}{E_4(\tau)} = 1 - 240q + 55440q^2 - 12793920q^3 + 2952385680q^4 \\ - 681306078240q^5 + 157221316739520q^6 + \ldots$$

Specifically, since there is a pole of $1/E_4$ at ρ, the radius of convergence around $i\infty$ must not include ρ. Specifically, it converges for $\mathrm{Im}(\tau) > \sqrt{3}/2$. Ramanujan conjectured and Bialek later proved that
$$\frac{1}{E_4(\tau)} = \sum_{n=0}^{\infty} \beta_n q^n$$
with (λ norms from $\mathcal{O}_{\mathbb{Q}(\rho)}$, with $\rho := e^{\frac{\pi i}{6}}$)
$$\beta_n := (-1)^n \frac{3}{E_6(\rho)} \sum_{(\lambda)} \sum_{(c,d)} \frac{h_{(c,d)}(n)}{\lambda^3} e^{\frac{\pi n\sqrt{3}}{\lambda}}. \tag{3.1}$$

Here (c, d) runs over "distinct" elements $c\rho + d$ with norm λ. Unlike the Fourier expansions of weakly holomorphic modular forms, the expansion does not converge everywhere. Finally, we let $h_{(1,0)}(n) := (-1)^n/2$, $h_{(2,1)}(n) := 1/2$, and for $\lambda \geq 7$ we let $(a, b) \in \mathbb{Z}^2$ be any solution to $ad - bc = 1$ and set
$$h_{(c,d)}(n) := \cos\left((ad + bc - 2ac - 2bd)\frac{\pi n}{\lambda} - 6\arctan\left(\frac{c\sqrt{3}}{2d - c}\right)\right).$$

Although it may not be clear from this definition, the $h_{(c,d)}$ are independent of the choice of a and b.

Hardy and Ramanujan showed that $1/E_6$ has a similar shape. It is thus natural to define ($\mathfrak{z} = \mathfrak{z}_1 + i\mathfrak{z}_2$, $\mathfrak{b}$ prim. ideals)

$$F_{k,j,r}(\tau,\mathfrak{z}) := \mathfrak{z}_2^{-j} \sum_{m=0}^{\infty} \sum_{\mathfrak{b}\subseteq\mathcal{O}_{\mathbb{Q}(\mathfrak{z})}}^{*} \frac{C_k(\mathfrak{b},m)}{N(\mathfrak{b})^{\frac{k}{2}-j}} (4\pi m)^r e^{\frac{2\pi m \mathfrak{z}_2}{N(\mathfrak{b})}} e^{2\pi i m\tau}.$$

Here C_k is similar to the $h_{(c,d)}$.

Hardy and Ramanujan and then Bialek, Berndt–Bialek, and Berndt–Bialek–Yee used the Circle Method to compute the coefficients. One has to be careful because the Fourier expansion doesn't converge everywhere. The calculations become increasingly difficult as the order of the poles increase. Berndt–Bialek–Yee studied forms with second-order poles in particular and the relevant calculations are quite long.

3.2. **Poincaré series.** Petersson attacked the problem differently. He constructing Poincaré series.

In particular, he computed the Fourier expansion of negative-weight meromorphic cusp forms with only simple poles (throughout this section, $k \geq 2$).

Theorem 3.1 (Petersson). *If f is a weight $2-2k<0$ meromorphic cusp form with only simple poles, then f has a Fourier expansion of the type*

$$f(\tau) \doteq \sum_{\substack{\mathfrak{z}\in\Gamma\backslash\mathbb{H}\\ \mathrm{ord}_{\mathfrak{z}}(f)=-1}} \mathrm{Res}_{\tau=\mathfrak{z}}(f(\tau)) \sum_{m=0}^{\infty} P_{2k,-m}(\mathfrak{z}) q^m.$$

Here $P_{2k,-m}$ are the weakly holomorphic Poincaré series

$$P_{2k,m}(z) := \sum_{M\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \varphi_m\big|_{2k} M(z),$$

where $k\in\mathbb{N}_{\geq 2}$ and for $m\in\mathbb{Z}$

$$\varphi_m(z) := e^{2\pi i m z}.$$

In the case of simple poles, it turns out that the formulas of Ramanujan are simply evaluations of Poincaré series at the points where poles occur. The norm of $c\mathfrak{z}+d$ is $\lambda = |c\mathfrak{z}+d|^2$ and $\mathrm{Im}(M\mathfrak{z}) = \frac{\mathfrak{z}_2}{|c\mathfrak{z}+d|^2}$, so the exponential becomes

$$e^{-2\pi i m \mathrm{Re}(M\mathfrak{z})} e^{2\pi m \mathrm{Im}(M\mathfrak{z})} = e^{-2\pi i m \mathrm{Re}(M\mathfrak{z})} e^{\frac{2\pi m \mathfrak{z}_2}{\lambda}};$$

we directly see the exponential which occurred in (3.1) (for $\mathfrak{z}=\rho$), while the other factor contributes to the cosine.

To obtain this result, let

$$H_{2k}(\mathfrak{z},\tau) := 2\pi i \sum_{M\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \frac{1}{1-e^{2\pi i(\tau-\mathfrak{z})}}\bigg|_{2k,\mathfrak{z}} M$$

The function H_{2k} is meromorphic in both $\mathfrak{z}$ and τ (with simple poles, unless the residue vanishes). It is a meromorphic modular form of weight $2k$ as a function of $\mathfrak{z}$, and Petersson furthermore showed that

$$H_{2k}(\mathfrak{z},\tau) \doteq \sum_{m=0}^{\infty} P_{2k,-m}(\mathfrak{z}) q^m.$$

Hence, in order to obtain Theorem 3.1, the main step is the following.

Theorem 3.2 (Petersson). *If f is a weight $2-2k$ meromorphic cusp form with simple poles, then there exist $\mathfrak{z}_1, \ldots, \mathfrak{z}_r$ and $c_\ell \in \mathbb{C}$*

$$f(\tau) = \sum_{\ell=1}^{r} c_\ell H_{2k}\left(\mathfrak{z}_\ell, \tau\right).$$

Indeed, Petersson obtained a converse theorem, classifying the choices of c_ℓ for which resulting form is indeed modular. The classification depends on a certain consistency condition from the Fourier coefficients and elliptic coefficients of cusp forms of weight $2k$. The elliptic expansion of a cusp form g around $\mathfrak{z}$ is ($X_{\mathfrak{z}}(\tau) := \frac{\tau - z}{\tau - \overline{\mathfrak{z}}}$)

$$g(\tau) = (\tau - \overline{\mathfrak{z}})^{-2k} \sum_{n \gg -\infty} a_{g,\mathfrak{z}}(n) X_{\mathfrak{z}}^{n}(\tau). \tag{3.2}$$

Namely, if f is a weight $2-2k$ meromorphic modular form and g is a weight $2k$ cusp form, then the product is a weight 2 meromorphic modular form. It turns out that the residue of fg at every point must vanish. This gives a necessary consistency condition, which also turns out to be sufficient for the sum of the H_{2k} functions to be modular.

Petersson's proof is useful for showing that the Fourier coefficients of forms with simple poles have the desired shape, but other Poincaré series are needed to address higher order poles. Let

$$H_{2k,\ell}(\mathfrak{z}, \tau) := 2\pi i \sum_{M \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \frac{\mathfrak{z}_2^{-\ell}}{1 - e^{2\pi i(\tau - \mathfrak{z})}} \bigg|_{2k,\mathfrak{z}} M$$

and

$$H_{2k,\ell}^{(r)}(\mathfrak{z}, \tau) := \frac{\partial^r}{\partial \tau^r} H_{2k,\ell}(\mathfrak{z}, \tau).$$

Lemma 3.3 (Bringmann–K.). *The Fourier coefficients of $H_{2k,\ell}^{(r)}$ agree with $F_{2k,\ell,r}$.*

Hence in order to obtain a formula for coefficients of meromorphic cusp forms with higher order poles, one needs a classification like Theorem 3.2 for higher order poles. Petersson has provided such a classification. For $\nu \leq -1$, define

$$Y_{2-2k,\nu}(\mathfrak{z}, \tau) := \frac{1}{(-\nu-1)!} \frac{\partial^{-\nu-1}}{\partial X_{\mathfrak{z}}(\alpha)^{-\nu-1}} \left[(\alpha - \overline{\mathfrak{z}})^{2k} H_{2k}(\alpha, \tau)\right]_{\alpha = \mathfrak{z}}.$$

The differential operator is naturally connected to the elliptic expansion (3.2).

Petersson showed the more general classification below.

Theorem 3.4 (Petersson). *If $f(\tau)$ is a meromorphic modular form with poles of order at most R, then f is a linear combination of $Y_{2-2k,\nu}(\mathfrak{z}, \tau)$ with $|\nu| \leq R$.*

By showing that $Y_{2-2k,\nu}$ is itself a linear combination of the $H_{2k,\ell}^{(r)}$ whenever $|\nu| \leq 3$, the following was shown.

Theorem 3.5 (Bringmann–K.). *If f is a negative-weight meromorphic cusp form with poles of order at most 3, then f is a linear combination of the $F_{m,j,r}$ functions.*

As the order of the poles increase, the proof becomes slightly more complicated, but is still less involved than the calculations with the Circle Method.

3.3. **Polar harmonic Maass forms.** The results of the previous section turn out to be much more general.

Theorem 3.6 (Bringmann–K.). *If f is a negative-weight meromorphic cusp form or quasi-modular form, then f is a linear combination of the $F_{m,j,r}$ functions.*

In order to obtain this more general result, one employs the theory of polar harmonic Maass forms. These are harmonic Maass forms which can also have singularities in the upper half-plane.

The key is to realize $H_{2k}(\tau, z)$ as the meromorphic part of a polar harmonic Poincaré Maass form $v^{2k-1}\Psi_{2k}(\mathfrak{z}, \tau)$, where we have the Poincaré series

$$\Psi_{2k}(\mathfrak{z}, \tau) := \sum_{M \in \mathrm{SL}_2(\mathbb{Z})} (\mathfrak{z} - \overline{\tau})^{-2k} X_\tau(\mathfrak{z})^{-1}\Big|_{2k,\mathfrak{z}} M.$$

The function $v^{2k-1}\Psi_{2k}(\mathfrak{z}, \tau)$ is a weight $2k$ meromorphic modular form as a function of $\mathfrak{z}$ and a weight $2-2k$ polar harmonic Maass form as a function of τ.

By repeatedly applying the *Maass raising operator*

$$R_{\kappa,\mathfrak{z}} := 2i\frac{\partial}{\partial \mathfrak{z}} + \frac{\kappa}{\mathfrak{z}_2}$$

in the $\mathfrak{z}$ variable, one obtains functions which are still harmonic as a function of τ, but have higher order poles. Specifically, define repeated raising by

$$R^n_{2k,\mathfrak{z}} := R_{2k+2n-2,\mathfrak{z}} \circ \cdots \circ R_{2k,\mathfrak{z}};$$

this sends a weight $2k$ object to a weight $2k+2n$ object. Using the image of $v^{2k-1}\Psi_{2k}(\mathfrak{z}, \tau)$ under these operators as a basis instead of Petersson's Y-functions, we have the following.

Proposition 3.7. *Every meromorphic cusp form of weight $2-2k$ is a linear combination of the functions*

$$R^n_{2k,\mathfrak{z}}\left(v^{2k-1}\Psi_{2k}(\mathfrak{z}, \tau)\right).$$

Furthermore, a linear combination

$$\sum_{\mathfrak{z} \in \mathbb{H}} \sum_{n=0}^{r_\mathfrak{z}} c_{n,\ell} R^n_{2k,\mathfrak{z}}\left(v^{2k-1}\Psi_{2k}(\mathfrak{z}, \tau)\right)$$

is a meromorphic cusp form if and only if its non-meromorphic part vanishes.

This classification is like Theorem 3.4, except that while he was investigating whether the a priori meromorphic linear combination was modular, the above proposition checks whether an a prior modular object is meromorphic. A key step to using the above classification to obtain formulas for all meromorphic modular forms is the following identity:

$$R^n_{2k,\mathfrak{z}}\left(\Psi_{2k}(\mathfrak{z}, \tau)\right) = \sum_{j=0}^{n} \frac{(2k+n-1)!}{(2k+n-1-j)!}\binom{n}{j}(-2i)^{n-j}\frac{\partial^{n-j}}{\partial \tau^{n-j}}\mathcal{H}_{2k+2n,j}(\mathfrak{z}, \tau),$$

where $\mathcal{H}_{2k,\ell}$ is a polar harmonic completion of $H_{2k,\ell}$. Namely, taking the meromorphic parts of each side, we have

$$R_{2k,\mathfrak{z}}^{n}\left(H_{2k}(\mathfrak{z},\tau)\right)=\sum_{j=0}^{n}\frac{(2k+n-1)!}{(2k+n-1-j)!}\binom{n}{j}(-2i)^{n-j}\frac{\partial^{n-j}}{\partial\tau^{n-j}}H_{2k+2n,j}(\mathfrak{z},\tau). \tag{3.3}$$

The right-hand sides have the correct shape to yield Theorem 3.6.

3.4. **Divisors of modular forms.** In the above, we assumed that $k\geq 2$, but it is natural to ask about the case $k=1$. For this, we extend to $\Gamma_0(N)$, since the case $N=1$ is not as interesting. In this case, it is natural to attempt to use Hecke's trick to extend the Poincaré series to weight 2. Due to the poles occurring in the upper half-plane and the fact that the Fourier expansion does not always exist, there is some care needed, however (specifically, one cannot analytically continue via the Fourier expansion because this would only give a continuation in part of $\mathbb{H}$). Define for $z,\tau\in\mathbb{H}$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$

$$P_{N,s}(\mathfrak{z},\tau):=\sum_{M\in\Gamma_0(N)}\frac{\varphi_s\left(M\mathfrak{z},\tau\right)}{j\left(M,\mathfrak{z}\right)^2\left|j\left(M,\mathfrak{z}\right)\right|^{2s}} \tag{3.4}$$

with $j(M,\tau):=(c\tau+d)$ and

$$\varphi_s(\mathfrak{z},\tau):=v^{1+s}(\mathfrak{z}-\tau)^{-1}(\mathfrak{z}-\overline{\tau})^{-1}\left|\mathfrak{z}-\overline{\tau}\right|^{-2s}.$$

One splits $P_{N,s}$ into three pieces. The first piece is those elements of Γ_∞, the second is

$$\sum_{M\in\Gamma_0(N)\setminus\Gamma_\infty}\frac{\varphi_s\left(M\mathfrak{z},\tau\right)-\varphi_s\left(M(i\infty),\tau\right)}{j\left(M,\mathfrak{z}\right)^2\left|j\left(M,\mathfrak{z}\right)\right|^{2s}},$$

and the third is

$$\sum_{M\in\Gamma_0(N)\setminus\Gamma_\infty}\frac{\varphi_s\left(M(i\infty),\tau\right)}{j\left(M,\mathfrak{z}\right)^2\left|j\left(M,\mathfrak{z}\right)\right|^{2s}}.$$

The first two pieces turn out to converge absolutely for $\mathrm{Re}(s)>-1$, while the last piece can be analytically continued via its Fourier expansion (as a function of $\mathfrak{z}$). The reason for this splitting is that the first two pieces have poles and hence their Fourier expansions do not converge everywhere, while the third piece does not converge absolutely, but its Fourier expansion exists everywhere and it can be analytically continued.

The analytic continuation to $s=0$ we write as $v\Psi_{2,N}(\mathfrak{z},\tau)$. We normalize to define

$$H_{N,\tau}^{*}(\mathfrak{z}):=:=-\frac{v}{2\pi}\Psi_{2,N}(\mathfrak{z},\tau).$$

This turns out to be harmonic both as a function of $\mathfrak{z}$ (in weight 2) and in τ (in weight 0). Its meromorphic part is a quasi-modular form (like E_2), and its non-meromorphic part is a constant multiple of $1/v$. As $\mathfrak{z}\to\tau$, it satisfies

$$H_{N,\tau}^{*}(\mathfrak{z})=\frac{\omega_{N,\tau}}{2\pi i}\frac{1}{\mathfrak{z}-\tau}+O(1), \tag{3.5}$$

where $e_{N,\tau}$ is half of the size of the stabilizer of τ in $\Gamma_0(N)$. At cusps $\sigma \neq i\infty$, one also defines

$$H^*_{N,\sigma}(\mathfrak{z}) := \lim_{\tau\to\sigma} H^*_{N,\tau}(\mathfrak{z}).$$

This exhibits a constant term as $\mathfrak{z}$ approaches the cusp σ. There is a similar definition at the cusp $i\infty$, although the limit does not directly exist.

Note that

$$\frac{D(f)}{f},$$

where $D := \frac{1}{2\pi i}\frac{\partial}{\partial\tau}$ (also written Θ), is also a quasi-modular form. To cancel off the terms which grow, it is natural to define

$$f^{\mathrm{div}}(\mathfrak{z}) := \sum_{\tau\in X_0(N)} \frac{1}{\omega_{N,\tau}}\mathrm{ord}_\tau(f) H^*_{N,\tau}(\mathfrak{z}). \tag{3.6}$$

The following then holds.

Theorem 3.8 (Bringmann–K.–Löbrich–Ono–Rolen)**.** *If $S_2(\Gamma_0(N))$ denotes the space of weight 2 cusp forms on $\Gamma_0(N)$, then*

$$f^{\mathrm{div}}(\mathfrak{z}) \equiv \frac{k}{4\pi\mathfrak{z}_2} - \frac{\Theta(f(\mathfrak{z}))}{f(\mathfrak{z})} \pmod{S_2(\Gamma_0(N))}.$$

From Theorem 3.8 and the asymptotic growth of the coefficients of $H^*_{N,\tau}$ (from a Ramanujan-like formula in weight 2), one can determine the points τ for which there are poles and zeros, and furthermore can determine the orders of these poles and zeros because those the the multiplicity with which $H_{N,\tau}$ occurs in f^{div}. This allows one to (numerically) compute the divisor of f given only the Fourier expansion (since $\Theta(f)$ can also be directly computed via the Fourier expansion).

4. Regularized Petersson inner products for meromorphic modular forms

For simplicity, we mostly assume integral weight and modularity for $\mathrm{SL}_2(\mathbb{Z})$ throughout.

4.1. Definition of the inner product.

4.1. Definition of the inner product. Considering the variables z and $\overline{z}$ as independent variables, note that for a weight $2k$ modular form $f(z)$, the function $\overline{f(z)}$ satisfies weight $2k$ modularity as a function of $\overline{z}$. Furthermore, writing $z = x+iy \in \mathbb{H}$, the function y^{2k} satisfies simultaneous weight $-2k$ modularity in both z and $\overline{z}$ because

$$\mathrm{Im}(Mz) = \mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\mathrm{Im}((az+b)(c\overline{z}+d))}{|cz+d|^2} = \frac{y}{|cz+d|^2},$$

where we used the fact that $ad - bc = 1$.

Petersson [14] then realized that, for functions f and g satisfying modularity for all $M \in \mathrm{SL}_2(\mathbb{Z})$, the function

$$f(z)\overline{g(z)}y^{2k}$$

is $\mathrm{SL}_2(\mathbb{Z})$-invariant. Moreover, the metric

$$\frac{dxdy}{y^2}$$

is also $\mathrm{SL}_2(\mathbb{Z})$-invariant. Hence the integral

$$\langle f, g\rangle := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(z)\overline{g(z)}y^{2k}\frac{dxdy}{y^2} \tag{4.1}$$

is well-defined whenever it converges absolutely. Using bounds for cusp forms (in particular, they exponentially decay as $y \to \infty$), one can show that the integral (4.1) converges absolutely for $f, g \in S_{2k}$. This exponential decay also suffices to show convergence when taking the inner product between $f \in S_{2K}$ and the Eisenstein series E_{2k}.

4.2. Petersson coefficient formula. The Petersson coefficient formula uses an explicit evaluation of the inner product to compute the Fourier coefficients of modular forms. To describe this result, we require the classical *Poincaré series* (see [16, 17])

$$P_{2k,m}(z) := \sum_{M\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \varphi_m\big|_{2k}M(z), \tag{4.2}$$

where $k \in \mathbb{N}_{\geq 2}$ and for $m \in \mathbb{Z}$

$$\varphi_m(z) := e^{2\pi imz}.$$

As discussed in the second talk, these converge locally and absolutely uniformly. For $m = 0$, the Poincaré series is precisely the Eisenstein series, while for $m > 0$ we have $P_{2k,m} \in S_{2k}$ and for $m < 0$ we have $P_{2k,m} \in M_{2k}^!$.

Theorem 4.1 (Petersson coefficient formula)**.** *If $f \in S_{2k}$ and $m \in \mathbb{N}$, then*

$$\langle f, P_{2k,m}\rangle = \frac{(2k-2)!}{(4\pi m)^{2k-1}}a_f(m).$$

Sketch of proof. Plugging in the definition (4.2) of the Poincaré series $P_{2k,m}$ and choosing a *fundamental domain* $\mathcal{F}$ for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ (a "nice" connected set of representatives $z\in\mathbb{H}$ of the orbits of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ under fractional linear transformations), we *unfold* the integral on the left-hand side by rewriting (formally, but this is valid because of the exponential decay of cusp forms towards the cusps)

$$\begin{aligned}\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(z) &\sum_{M\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \frac{\overline{\varphi_m(Mz)}}{(c\overline{z}+d)^{2k}} y^{2k}\frac{dxdy}{y^2}\\ &= \sum_{M\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}\int_{\mathcal{F}} f(Mz)\varphi_m(Mz)\mathrm{Im}(Mz)^{2k}\frac{dxdy}{y^2}\\ &= \sum_{M\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}\int_{M\mathcal{F}} f(z)\varphi_m(z)y^{2k}\frac{dxdy}{y^2} = \int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{\varphi_m(z)}y^{2k}\frac{dxdy}{y^2}.\end{aligned}\tag{4.3}$$

Since the fundamental domain for $\Gamma_\infty\backslash\mathbb{H}$ is very simple, this unfolding argument results in the double integral

$$\int_0^\infty\int_0^1 f(z)\overline{\varphi_m(z)}y^{2k}\frac{dxdy}{y^2}.$$

The integral over x essentially picks off the mth coefficient and then explicitly computing the integral over y yields the claim. □

4.3. **Orthogonal splitting.** The inner product on S_{2k} is positive-definite. Hence, by the Gram-Schmidt process, one can construct an orthonormal basis. A particular choice of the basis elements turns out to be very natural. Namely, one can obtain an orthonormal basis of Hecke eigenforms. Why are these orthogonal? Since the Hecke operators are Hermitian, we have

$$\lambda_f(n)\langle f,g\rangle = \langle\lambda_f(n)f,g\rangle = \langle f|_{2k}T_n,g\rangle = \langle f,g|_{2k}T_n\rangle = \langle f,\lambda_g(n)g\rangle = \lambda_g(n)\langle f,g\rangle.$$

Since $\lambda_f(n)\neq\lambda_g(n)$, this leads to a contradiction if $\langle f,g\rangle\neq 0$. We thus conclude that f and g are orthogonal to each other. Hence the splitting of S_{2k} into eigenspaces precisely yields the orthogonal splitting, with the orthonormal basis given by the Hecke eigenforms normalized such that $\|f\|^2=1$.

We note that the other normalization $a_f(1)=1$ is also natural. Under this normalization (and appropriately normalizing the Hecke operators), the coefficients $a_f(n)$ and the eigenvalues $\lambda_f(n)$ coincide. This realization "de-mystifies" the coefficients of the Hecke eigenforms and plays an important role in understanding Fourier expansions.

4.4. **Inner products for weakly holomorphic modular forms.**

4.4.1. *The regularization of Petersson, Harvey–Moore, and Borcherds and its extension.* For $f,g\in M_{2k}^!$, the integral (4.1) generally diverges. Petersson established a Cauchy principal value for the integral as a partial solution to this problem. Instead of integrating over the full fundamental domain, we integrate over a *cut-off fundamental domain* whose closure does not include the cusp on the boundary of the chosen fundamental domain. In our case, the cusp is $i\infty$ and the cut-off fundamental domain is given by

$$\mathcal{F}_T := \left\{ z \in \mathbb{H} : |z| \geq 1,\ y \leq T,\ -\frac{1}{2} \leq x \leq \frac{1}{2} \right\}.$$

For $f, g \in M_{2k}^!$, Petersson then defined the regularized inner product (see [15])

$$\langle f, g \rangle := \lim_{T \to \infty} \int_{\mathcal{F}_T} f(z)\overline{g(z)} y^{2k} \frac{dxdy}{y^2}. \tag{4.4}$$

The key to the above regularization is that it essentially gives an ordering to the integrals over x and y.

This construction was further independently rediscovered and extended by Harvey–Moore [11] and Borcherds [2] by multiplying the integrand by y^s for some $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$ and then taking the constant term of the Laurent expansion of the meromorphic continuation (in s) at $s = 0$.

One can use the regularized inner product to show that for $m < 0$ the Poincaré series $P_{2k,m}$, defined in (4.2), is orthogonal to cusp forms. This was shown by Petersson in a much more general setting in [15, Satz 4].

The regularization of Petersson/Harvey–Moore/Borcherds does not always converge, however. In particular, Petersson found a necessary and sufficient condition for his regularization (4.4)to converge (see [15, Satz 1]) and Petersson norms once again pose a problem, as they did for the Eisenstein series. This problem has been recently resolved by Bringmann, Diamantis, and Ehlen [3], who were able to extend the regularization in a way so that the inner product $\langle f, g \rangle$ is well-defined and finite for all $f, g \in M_{2k}^!$. The basic idea is to multiply the integrand by an invariant function $h_s(z)$ which satisfies $h_0(z) = 1$ and such that the integral converges for $\mathrm{Re}(s)$ sufficiently large. One then takes the analytic continuation to $s = 0$ as the definition of the inner product. We do not give any of the technical details here, but the reader is encouraged to look at [3, Section 3, and in particular Theorem 3.2].

4.4.2. *Theta lifts.* The inner product has been used by many authors (for example, in [2] and [6]) to obtain theta lifts from modular forms of one type to modular forms of another type. To give a rough idea, one defines a two-variable theta function $\Theta(z, \tau)$ which is modular in both variables (one calls this function the *theta kernel*), but which satisfies a different kind of modularity in each variable (for example, suppose that it satisfies weight $2k$ modularity as a function of z and weight $k + 1/2$ modularity as a function of τ). Taking the inner product in one variable against another function f satisfying the same type of modularity then yields a new function in the other variable satisfying the other type of modularity. In other words, in the example above, if f satisfies weight $2k$ modularity, then

$$\Phi(f)(\tau) := \langle \Theta(\cdot, \tau), f \rangle$$

satisfies weight $k + 1/2$ modularity. This yields a *theta lift* Φ from weight $2k$ modular forms to weight $k + 1/2$ modular forms. The example illustrated above is Shintani's construction [19] of his lift from integral weight to half-integral weight modular forms and the lift in the opposite direction can be shown to be one of Shimura's lifts [18] from half-integral weight to integral weight (see [13] and [12] for two alternative options for the theta kernel).

Lifts from "simpler" spaces with special properties often yield strange or exceptional modular forms which can be used to understand or narrow down conjectures that are often precisely false on the image or pre-image of such lifts. For example, the Shimura lift generally sends cusp forms to cusp forms, but there is an exceptional class of forms known as unary theta functions in weight $3/2$ which are cusp forms but whose image under the Shimura lift is an Eisenstein series. These unary theta functions are also counter-examples to the *Ramanujan–Petersson conjecture*, which states that the coefficients of weight $\kappa \in \frac{1}{2}\mathbb{Z}$ cusp forms f satisfy

$$|a_f(n)| \ll_{f,\varepsilon} n^{\frac{\kappa-1}{2}+\varepsilon}.$$

The coefficients of the unary theta functions grow like $n^{1/2}$, contradicting the conjecture in this wide of generality. However, for integral weight cusp forms $f \in S_{2k}$, the conjecture is a celebrated result of Deligne [7] and it is conjectured that the Ramanujan–Petersson conjecture holds in half-integral weight as long as f is orthogonal to unary theta functions.

4.4.3. *Computation of the inner product by the Brunier–Funke pairing.* For $f, g \in M_{2k}^!$, we next describe a way to compute the inner product between these two forms. Let G be a harmonic Maass form for which

$$\xi_{2-2k}(G) = g.$$

The inner product between f and g is then given by the *Bruinier–Funke pairing* between the function G and f, given by

$$\{f, G\} := \sum_{n\in\mathbb{Z}} a_f(-n) a_G^+(n), \tag{4.5}$$

where $a_G^+(n)$ is the nth coefficient of the holomorphic part of the Fourier expansion. In particular, we have

$$\langle f, g\rangle = \{f, G\}. \tag{4.6}$$

The pairing is useful for computing inner products because only finitely many terms in (4.5) are non-zero.

Roughly speaking, the pairing is shown by using Stokes Theorem to evaluate the integral instead of the unfolding method described in Section 4.2. The pre-image G of g under ξ_{2-2k} naturally appears in this context.

4.5. **Inner products for meromorphic modular forms.** We would now like to define an inner product on arbitrary meromorphic modular forms $f, g \in \mathcal{S}_{2k}$. However, an arbitrary meromorphic modular form $f \in \mathcal{S}_{2k}$ may be decomposed into two pieces, one of which only has poles at the cusps (i.e., it is in $M_{2k}^!$) and one of which only has poles in the upper half-plane (vanishing towards all cusps); we call forms of the second type weight $2k$ *meromorphic cusp forms* and denote the subspace of such forms by $\mathbb{S}_{2k}$. It thus essentially suffices to consider inner products between forms $f, g \in \mathbb{S}_{2k}$ (technically, we also have to take inner products between forms $f \in M_{2k}^!$ and $g \in \mathbb{S}_{2k}$, but hybrid approaches for the regularizations will work in full generality and we ignore the details here).

4.5.1. *Regularization of Petersson.* The idea that Petersson used to generalize (4.1) is very similar to the idea used in the regularization (4.4). Instead of cutting off the fundamental domain away from $i\infty$, one cuts out small neighborhoods around each pole $\mathfrak{z}$ of f or g and then shrinks the hyperbolic volume of the neighborhoods to zero in a limit. In particular, for $\mathfrak{z} \in \mathbb{H}$ define the ball

$$\mathcal{B}_\varepsilon(\mathfrak{z}) := \{z \in \mathbb{H} : r_{\mathfrak{z}}(z) < \varepsilon\},$$

where $r_{\mathfrak{z}}(z) := |X_{\mathfrak{z}}(z)|$ with

$$X_{\mathfrak{z}}(z) := \frac{z - \mathfrak{z}}{z - \overline{\mathfrak{z}}}.$$

The functions $r_{\mathfrak{z}}(z)$ are naturally connected to the hyperbolic distance $d(z, \mathfrak{z})$ between z and $\mathfrak{z} = \mathfrak{z}_1 + i\mathfrak{z}_2$ in $\mathbb{H}$ via the formula

$$r_{\mathfrak{z}}(z) = \tanh\left(\frac{d(z, \mathfrak{z})}{2}\right).$$

Let $[z_1], \ldots, [z_r] \in \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ be the distinct $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of all of the poles of $f, g \in \mathbb{S}_{2k}$ and choose a fundamental domain $\mathcal{F}^*$ such that all z_ℓ lie in the interior of $\Gamma_{z_\ell}\mathcal{F}^*$, where $\Gamma_{\mathfrak{z}}$ is the stabilizer of $\mathfrak{z}$ in $\mathrm{PSL}_2(\mathbb{Z})$. Petersson's regularized inner product is then defined by

$$\langle f, g\rangle := \lim_{\varepsilon_1, \ldots, \varepsilon_r \to 0^+} \int_{\mathcal{F}^* \backslash \left(\bigcup_{\ell=1}^r \mathcal{B}_{\varepsilon_\ell}(z_\ell)\right)} f(z)\overline{g(z)} y^{2k} \frac{dxdy}{y^2}. \tag{4.7}$$

A necessary and sufficient condition for the convergence of the regularization (4.7) is given by Petersson in [15, Satz 1]. Furthermore, certain Poincaré series related to the elliptic expansions (Petersson proved an elliptic coefficient formula as well; cf. [15, Satz 9]) with poles in the upper half-plane were also shown to be orthogonal to cusp forms in [15, Satz 7]. Once again, Petersson's necessary and sufficient condition implies that his regularization diverges in particular when evaluating Petersson norms for elements of $\mathbb{S}_{2k}$ which are not cusp forms.

4.5.2. *A new regularization.* Since Petersson's regularization still sometimes diverges, one requires a further regularization; we recall the construction from [5]. The idea is similar to the regularization of Bringmann–Diamantis–Ehlen; roughly speaking, the integrand in (4.1) is multiplied by an $\mathrm{SL}_2(\mathbb{Z})$-invariant function $H_s(\tau)$ which removes the poles of the integrand whenever $\mathrm{Re}(s)$ is sufficiently large. We then take the constant term of the Laurent expansion around $s = 0$ to be our regularization. To be more precise, let $[z_1], \ldots, [z_r] \in \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ be the distinct $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of all of the poles of f and g and define

$$\langle f, g\rangle := \mathrm{CT}_{s=0}\left(\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(z) H_s(z)\overline{g(z)} y^{2k} \frac{dxdy}{y^2}\right), \tag{4.8}$$

where

$$H_s(z) = H_{s_1, \ldots, s_r, z_1, \ldots, z_r}(z) := \prod_{\ell=1}^r h_{s_\ell, z_\ell}(z).$$

Here

$$h_{s_\ell, z_\ell}(z) := r_{z_\ell}^{2s_\ell}(Mz),$$

with $M \in \mathrm{SL}_2(\mathbb{Z})$ chosen such that $Mz \in \mathcal{F}^*$. Moreover $\mathrm{CT}_{s=0}$ denotes the constant term in the Laurent expansion around $s_1 = s_2 = \cdots = s_r = 0$ of the meromorphic continuation (if existent).

In the same sense that the results in [3] may be viewed as an analytic definition for a regularized inner product satisfying the Bruinier–Funke pairing for arbitrary $f, g \in M^!_{2k}$, the above regularization may be viewed as an analytic definition for a regularized integral giving a similar pairing for all $f, g \in \mathbb{S}_{2k}$. However, instead of defining the pairing via the Fourier expansions, the pairing is defined via the elliptic expansions of f and a weight $2-2k$ *polar harmonic Maass form* (i.e., a harmonic Maass form with singularities in the upper half-plane) G which is a pre-image of g under the ξ-operator. To describe the pairing, the *elliptic expansion* of $f \in \mathbb{S}_{2k}$ around $\mathfrak{z} \in \mathbb{H}$ is given by

$$f(z) = (z - \overline{\mathfrak{z}})^{-2k} \sum_{n \gg -\infty} a_{f,\mathfrak{z}}(n) X_{\mathfrak{z}}^n(z). \tag{4.9}$$

For the polar harmonic Maass form G, we again denote the coefficients of its meromorphic part (i.e., of the form in (4.9)) by $a^+_{G,\mathfrak{z}}(n)$.

Denoting $\mathfrak{z}_2 := \mathrm{Im}(\mathfrak{z})$ and writing $\omega_{\mathfrak{z}}$ for the size of the stabilizer $\Gamma_{\mathfrak{z}}$ of $\mathfrak{z}$ in $\mathrm{PSL}_2(\mathbb{Z})$, the pairing is given by (see [4, Proposition 6.1])

$$\{f, G\} := \sum_{\mathfrak{z} \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \frac{\pi}{\mathfrak{z}_2 \omega_{\mathfrak{z}}} \sum_{n \in \mathbb{Z}} a_{f,\mathfrak{z}}(n) a^+_{G,\mathfrak{z}}(-n-1). \tag{4.10}$$

It is again important to emphasize that the pairing gives a formula for the inner product with only finitely many coefficients in (4.10) non-zero. In comparison, Petersson evaluated his inner product (4.4) (resp. (4.7)) on [15, pages 42–43] via the Fourier (res. elliptic) coefficients of the forms f and g themselves, but his evaluation is given as an infinite sum, so one can only obtain an approximation for the inner product by computing the Fourier (resp. elliptic) coefficients. In other words, Petersson's constructions are better in the sense that they are given in terms of the coefficients of the original functions, while one is required to introduce new functions to determine (4.5) and (4.10), but the sums in these pairings are instead finite.

4.5.3. *Higher Greens functions.* The regularization (4.8) was used in [5] to compute the inner product between

$$f_Q(z) = f_{k,-D,[Q]}(z) := D^{\frac{k}{2}} \sum_{\mathcal{Q} \in [Q]} \mathcal{Q}(z,1)^{-k}$$

for positive-definite integral binary quadratic forms Q of discriminant $-D$. These are weight $2k$ meromorphic modular forms which have poles of order k at the unique zero τ_Q of Q in $\mathbb{H}$. The evaluation of the inner product between two such functions is done by again using Stokes Theorem to rewrite the inner product as the pairing (4.10) in terms of the elliptic coefficients of f_Q and the elliptic coefficients of the meromorphic part of a polar harmonic Maass form $\mathcal{G}_Q$ associated with f_Q via the ξ-operator. It then remains to explicitly compute the elliptic coefficients occurring in (4.10).

In particular, choosing two such binary quadratic forms Q and $\mathcal{Q}$, the inner product between f_Q and $f_{\mathcal{Q}}$ is related to the *higher Green's function* $G_k : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$, which is uniquely characterized by the following properties:

(1) G_k is a smooth real-valued function on $\mathbb{H}\times\mathbb{H}\setminus\{(z,\gamma z)|\gamma\in\Gamma, z\in\mathbb{H}\}$.
(2) For $\gamma_1,\gamma_2\in\Gamma$, we have $G_k(\gamma_1 z,\gamma_2\mathfrak{z})=G_k(z,\mathfrak{z})$.
(3) Denoting $\Delta_{0,z}:=-4y^2\frac{\partial}{\partial z}\frac{\partial}{\partial\overline{z}}$, we have

$$\Delta_{0,z}\left(G_k\left(z,\mathfrak{z}\right)\right)=\Delta_{0,\mathfrak{z}}(G_k\left(z,\mathfrak{z}\right))=k(1-k)G_k\left(z,\mathfrak{z}\right).$$

(4) As $z\to\mathfrak{z}$

$$G_k(z,\mathfrak{z})=2\omega_{\mathfrak{z}}\log\left(r_{\mathfrak{z}}(z)\right)+O(1).$$

(5) As z approaches a cusp, $G_k(z,\mathfrak{z})\to 0$.

These higher Green's functions have a long history, appearing as special cases of the resolvent kernel studied by Fay [8] and investigated thoroughly by Hejhal in [10], for example. Gross and Zagier [9] conjectured that their evaluations at CM-points are essentially logarithms of algebraic numbers, which has been since proven in a number of cases. To state the connection with inner products, let $\beta(a,b):=\int_0^1 t^{a-1}(1-t)^{b-1}dt$ be the beta function, and let $\mathcal{Q}_{-D}$ denote the set of positive-definite integral binary quadratic forms of discriminant $-D<0$. Evaluating the elliptic coefficients in (4.10) for f_Q and $\mathcal{G}_Q$ then yields the following theorem.

Theorem 4.2 (Theorem 1.5 of [5]). *For $Q\in\mathcal{Q}_{-D_1}$ and $\mathcal{Q}\in\mathcal{Q}_{-D_2}$ ($-D_1,-D_2<0$ discriminants) with $[\tau_Q]\neq[\tau_{\mathcal{Q}}]$, we have*

$$\langle f_{\mathcal{Q}},f_Q\rangle=-\frac{\pi(-4)^{1-k}}{(2k-1)\beta(k,k)}\frac{G_k(\tau_{\mathcal{Q}},\tau_Q)}{\omega_{\tau_{\mathcal{Q}}}\omega_{\tau_Q}}.$$

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Department of Mathematics, University of Hong Kong, Pokfulam, Hong Kong
E-mail address: bkane@maths.hku.hk